Duration : 144 minutes

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Linear Algebra Exam

Common part

Fall 2017

Answers

For the **multiple choice** questions, we give

$+3$ points if your answer is correct,

0 points if you give no answer or more than one,

-1 if your answer is incorrect.

The notation and terminology of this exam are those used in the exercise sheets and the lectures of the course Linear Algebra given during the Fall semester 2017.

Notation

– For a matrix A, a_{ij} denotes the entry of A in row i and column j.
– For a vector $\boldsymbol{x}$, x_i denotes the i-the coordinate of $\boldsymbol{x}$.
– I_m denotes the $m \times m$ identity matrix.
– $\mathbb{P}_n$ is the vector space of polynomials of degree less than or equal to n.
– The inner product of vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ is defined as $\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{x}^T \boldsymbol{y}$.
– The length of a vector $\boldsymbol{x} \in \mathbb{R}^n$ is defined as $\|\boldsymbol{x}\| = \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}}$.

Multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 : If A is a 7×6 matrix whose last three columns are linearly dependent, then

- [ ] $\operatorname{rank} A \geq 2$
- [x] $\operatorname{rank} A \leq 5$
- [ ] $\operatorname{rank} A = 3$
- [ ] $\operatorname{rank} A = 4$

Question 2 : Let h be a real parameter. Let

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ h \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} h \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix} 1 \\ h \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{v}_4 = \begin{pmatrix} 0 \\ 1 \\ h \\ 1 \end{pmatrix}.$$

Then $\boldsymbol{v}_1$ belongs to the subspace of $\mathbb{R}^4$ generated by $\{\boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ if and only if

- [x] $h = -2$, $h = 0$, or $h = 2$
- [ ] $h = 0$ or $h = 2$
- [ ] $h = 0$
- [ ] $h = -2$ or $h = 2$

Question 3 : Let A be an $n \times n$ matrix and let P be an $n \times n$ matrix such that each column of P is an eigenvector of A. Then, it is always true that

- [x] $AP = PD$ for a diagonal matrix D
- [ ] P is invertible and PAP^{-1} is a diagonal matrix
- [ ] P is invertible and $P^{-1}AP$ is a diagonal matrix
- [ ] $PA = DP$ for a diagonal matrix D

Question 4 : Let $\boldsymbol{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right\}$.

The orthogonal projection of $\boldsymbol{v}$ onto W is

- ☐ $\begin{pmatrix} 3 \\ 5/2 \\ 1/2 \end{pmatrix}$
- ■ $\begin{pmatrix} 2 \\ 3/2 \\ 5/2 \end{pmatrix}$
- ☐ $\begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$
- ☐ $\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$

Question 5 : Let A be an $m \times n$ matrix. Suppose that there are vectors $\boldsymbol{v} \in \mathbb{R}^n$ and $\boldsymbol{w} \in \mathbb{R}^k$ and an $m \times k$ matrix B such that $A\boldsymbol{v} = B\boldsymbol{w}$. Then it is always true that

- ☐ $n = k$
- ■ $B\boldsymbol{w} \in \text{Col}(A)$
- ☐ the system $A\boldsymbol{x} = \boldsymbol{b}$ has at least one solution for every $\boldsymbol{b} \in \mathbb{R}^m$
- ☐ $\boldsymbol{w} = B^{-1}A\boldsymbol{v}$

Question 6 : Let $\mathcal{C} = \{1, t, t^2\}$ be the standard basis of $\mathbb{P}_2$ and let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be the linear transformation defined by

$$T(a + bt + ct^2) = a + b(t-1) + c(t-1)^2 \qquad \text{for all } a, b, c \in \mathbb{R}.$$

Let $M = [T]_{\mathcal{C} \to \mathcal{C}}$ be the matrix that represents T in the basis $\mathcal{C}$; in other words, the matrix M such that $\big[T(p)\big]_{\mathcal{C}} = M\big[p\big]_{\mathcal{C}}$ for all $p \in \mathbb{P}_2$. Then

- ☐ $M = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$
- ☐ $M = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$
- ■ $M = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$
- ☐ $M = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$

Question 7 : Let

$$A = \begin{pmatrix} 1 & 0 & -3 \\ -2 & 2 & 1 \\ 3 & -8 & 7 \end{pmatrix}.$$

Compute the LU factorization of A (using only the elementary row operation that adds a multiple of one row to another row below it). Then the matrix L equals

- ☐ $L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 4 & 1 \end{pmatrix}$
- ■ $L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -4 & 1 \end{pmatrix}$
- ☐ $L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -8 & 1 \end{pmatrix}$
- ☐ $L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 8 & 1 \end{pmatrix}$

Question 8 : Let A be an $m \times n$ matrix and $\boldsymbol{b} \in \mathbb{R}^m$. Let $\boldsymbol{c} = \operatorname{proj}_{\operatorname{Col}(A)} \boldsymbol{b}$. Then, it is always true that

- [ ] the least-squares solution of the equation $A\boldsymbol{x} = \boldsymbol{b}$ is $A^{-1}\boldsymbol{c}$
- [ ] the equation $A\boldsymbol{x} = \boldsymbol{b}$ has no solution
- [x] every solution of $A\boldsymbol{x} = \boldsymbol{c}$ is a least-squares solution of $A\boldsymbol{x} = \boldsymbol{b}$
- [ ] the equation $A\boldsymbol{x} = \boldsymbol{c}$ has a unique solution

Question 9 : Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation defined by

$$T\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}\right) = \begin{pmatrix} x_1 + 2x_2 + x_4 \\ 3x_1 + 6x_2 + x_3 \\ x_1 + 2x_2 + x_3 - 2x_4 \\ 3x_3 - 9x_4 \end{pmatrix}.$$

Then

- [ ] $\operatorname{Ker} T = \operatorname{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ -9 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \\ 1 \end{pmatrix}\right\}$
- [ ] $\operatorname{Ker} T = \operatorname{Span}\left\{\begin{pmatrix} 9 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}\right\}$
- [x] $\operatorname{Ker} T = \operatorname{Span}\left\{\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 3 \\ 1 \end{pmatrix}\right\}$
- [ ] $\operatorname{Ker} T = \operatorname{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -3 \end{pmatrix}\right\}$

Question 10 : Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation from the previous question. Then

- [ ] $\operatorname{Im} T = \operatorname{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -3 \end{pmatrix}\right\}$
- [ ] $\operatorname{Im} T = \operatorname{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right\}$
- [ ] $\operatorname{Im} T = \operatorname{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 1 \\ 0 \end{pmatrix}\right\}$
- [x] $\operatorname{Im} T = \operatorname{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ -2 \\ -9 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 3 \end{pmatrix}\right\}$

Question 11 : Let A and B be two invertible $n \times n$ matrices. Then the number

$$\det(A^{-1}) \det(A + B) \det(B^{-1})$$

- [ ] equals 2
- [ ] equals $\det(B^{-1}) + \det(A^{-1})$
- [ ] is not defined because the matrix $A + B$ is not necessarily invertible
- [x] equals $\det(B^{-1} + A^{-1})$

Question 12 : Let A be a 3×3 matrix such that

$$\det(A - \lambda I_3) = -(\lambda - 1)^2(\lambda + 1).$$

Which of the following statements are always true?

(a) the matrix A is diagonalizable,

(b) the matrix A is invertible,

(c) the eigenspace of the eigenvalue $\lambda = 1$ has dimension 2

- ☐ only (a) and (c)
- ☐ all three statements
- ■ only (b)
- ☐ none of the three statements

Question 13 : Let B be an $n \times m$ matrix such that $B^T B = I_m$ and let A be the $n \times n$ matrix defined by $A = I_n - 2BB^T$. Which of the following statements are always true?

(a) $A^T = A$, **(b)** $A^2 = A$, **(c)** $A^T A = I_n$, **(d)** $A = -I_n$.

- ☐ (a), (b), (c), and (d)
- ■ only (a) and (c)
- ☐ only (a) and (d)
- ☐ only (a) and (b)

Question 14 : Let $A = \begin{pmatrix} 1 & 3 \\ 0 & 0 \\ -2 & -1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix}$.

Then the least-squares solution $\widehat{\boldsymbol{x}} = \begin{pmatrix} \widehat{x}_1 \\ \widehat{x}_2 \end{pmatrix}$ of the equation $A\boldsymbol{x} = \boldsymbol{b}$ satisfies

- ☐ $\widehat{x}_1 = 5$
- ■ $\widehat{x}_1 = -4$
- ☐ $\widehat{x}_2 = 5$
- ☐ $\widehat{x}_2 = -4$

Question 15 : The matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix}$$

- ☐ is not invertible
- ☐ is invertible, and the coefficient b_{14} of its inverse $B = A^{-1}$ is equal to $\frac{1}{2}$
- ☐ is invertible, and the coefficient b_{14} of its inverse $B = A^{-1}$ is equal to 2
- ■ is invertible, and the coefficient b_{14} of its inverse $B = A^{-1}$ is equal to 1

Question 16 : Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} 2x_2 + 3x_3 \\ 4x_1 - 3x_2 + 2x_3 \\ 5x_1 - 6x_3 \end{pmatrix}. \quad \text{Let } \mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Let $M = [T]_{\mathcal{B}\to\mathcal{B}}$ be the matrix that represents T in the basis $\mathcal{B}$; in other words, the matrix M such that $\left[T(\boldsymbol{x})\right]_{\mathcal{B}} = M\left[\boldsymbol{x}\right]_{\mathcal{B}}$ for all $\boldsymbol{x} \in \mathbb{R}^3$. Then

- [ ] $M = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 6 & -3 \\ 5 & -1 & 0 \end{pmatrix}$
- [ ] $M = \begin{pmatrix} -3 & 5 & 4 \\ 4 & 1 & -2 \\ 2 & 0 & -5 \end{pmatrix}$
- [x] $M = \begin{pmatrix} -3 & 4 & 2 \\ 5 & -1 & 0 \\ 4 & 2 & -5 \end{pmatrix}$
- [ ] $M = \begin{pmatrix} 2 & 1 & 5 \\ 3 & 6 & -1 \\ 2 & -3 & 0 \end{pmatrix}$

Question 17 : Let

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -4 & 1 & 0 \\ 2 & 2 & 3 \end{pmatrix}.$$

The eigenvalues of A are

- [x] 1 and 4
- [ ] 1 and 2
- [ ] 1, 2, and 3
- [ ] 1 and 3

Question 18 : Let A be the matrix from the previous question and let E_1 be the eigenspace of the eigenvalue $\lambda = 1$ of A. Then

- [x] $E_1 = \text{Span}\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}$
- [ ] $E_1 = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$
- [ ] $E_1 = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}$
- [ ] $E_1 = \text{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

Question 19 : The matrix

$$B = \begin{pmatrix} 0 & x & 1 & 2 \\ 1 & 0 & 0 & x \\ -1 & 1 & x & 0 \\ 2 & -1 & 0 & x \end{pmatrix}$$

is invertible if and only if

- [ ] $x \neq 0\,,\ x \neq -\sqrt{3}\,,\ \text{and}\ x \neq \sqrt{3}$
- [ ] $x \neq 0\,,\ x \neq -2\,,\ \text{and}\ x \neq 2$
- [x] $x \neq 0\,,\ x \neq -\sqrt{2}\,,\ \text{and}\ x \neq \sqrt{2}$
- [ ] $x \neq 0\,,\ x \neq -\frac{1}{\sqrt{2}}\,,\ \text{and}\ x \neq \frac{1}{\sqrt{2}}$

Question 20 : Which one of the following bases of $\mathbb{R}^3$ is orthonormal?

■ $\left\{ \begin{pmatrix} \sqrt{3}/3 \\ \sqrt{3}/3 \\ \sqrt{3}/3 \end{pmatrix}, \begin{pmatrix} \sqrt{2}/2 \\ 0 \\ -\sqrt{2}/2 \end{pmatrix}, \begin{pmatrix} \sqrt{6}/6 \\ -\sqrt{6}/3 \\ \sqrt{6}/6 \end{pmatrix} \right\}$

☐ $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$

☐ $\left\{ \begin{pmatrix} \sqrt{3}/3 \\ \sqrt{3}/3 \\ \sqrt{3}/3 \end{pmatrix}, \begin{pmatrix} 2\sqrt{5}/5 \\ \sqrt{5}/5 \\ 0 \end{pmatrix}, \begin{pmatrix} \sqrt{6}/6 \\ -\sqrt{6}/3 \\ \sqrt{6}/6 \end{pmatrix} \right\}$

☐ $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$

Question 21 : Let A and B be two diagonalizable $n \times n$ matrices such that each eigenspace of B is contained in an eigenspace of A. Then

☐ AB is diagonalizable if and only if A and B have the same eigenvalues

☐ AB is never diagonalizable

■ AB is always diagonalizable

☐ AB is diagonalizable if and only if A and B have the same eigenspaces

Question 22 : Let h be a real parameter, and let

$$A = \begin{pmatrix} 1 & 7 & 3 \\ 2 & 14 & 1 \\ -1 & -7 & 1 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{pmatrix} h+11 \\ 6 \\ h-1 \end{pmatrix}.$$

Then the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$

☐ has the unique solution $\begin{pmatrix} 7 \\ -1 \\ 6 \end{pmatrix}$ if $h \neq 7$

■ has no solution if $h \neq 7$

☐ has the unique solution $\begin{pmatrix} 7 \\ -1 \\ 6 \end{pmatrix}$ if $h = 7$

☐ has the solution $\begin{pmatrix} 1 \\ -7 \\ 6 \end{pmatrix}$ if $h = 7$

Question 23 : Let

$$\mathcal{E} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad \mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

be two bases of $\mathbb{R}^3$. Let $P_{\mathcal{E}\to\mathcal{B}}$ be the change of basis matrix from $\mathcal{E}$ to $\mathcal{B}$; in other words, we have $\left[\boldsymbol{x}\right]_{\mathcal{B}} = P_{\mathcal{E}\to\mathcal{B}}\left[\boldsymbol{x}\right]_{\mathcal{E}}$ for all $\boldsymbol{x} \in \mathbb{R}^3$. Then

- ☐ $P_{\mathcal{E}\to\mathcal{B}} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}$
- ■ $P_{\mathcal{E}\to\mathcal{B}} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$
- ☐ $P_{\mathcal{E}\to\mathcal{B}} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$
- ☐ $P_{\mathcal{E}\to\mathcal{B}} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$

Question 24 : How many of the following four subsets of $\mathbb{P}_3$ are subspaces of $\mathbb{P}_3$?

$\{p \in \mathbb{P}_3 \mid p(0) = 2\,,\ p(2) = 0\}$, $\quad \{p \in \mathbb{P}_3 \mid p'(t) = 0 \text{ for every } t \in \mathbb{R}\}$,

$\{p \in \mathbb{P}_3 \mid p(t) = 2a - at^3 \text{ with } a \in \mathbb{R}\}$, $\quad \{p \in \mathbb{P}_3 \mid p(t) = ct^2 - c^2t \text{ with } c \in \mathbb{R}\}$

- ■ 2
- ☐ 1
- ☐ 3
- ☐ 4